On general (α,β)-metrics with vanishing Douglas curvature

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Abstract

In this paper, we study a class of Finsler metrics called general (α,β)-metrics, which are defined by a Riemannian metric α and a 1-form β. We find an equation which is necessary and sufficient condition for such Finsler metric to be a Douglas metric. By solving this equation, we obtain all of general (α,β)-metrics with vanishing Douglas curvature under certain condition. Many new non-trivial examples are explicitly constructed.

1 Introduction

In Finsler geometry, one of important projective invariants is Douglas curvature, which was introduced by J. Douglas [4]. If two Finsler metrics F and $\tilde{F}$ are projectively equivalent, then they have the same Douglas curvature. The Douglas curvature always vanishes for Riemannian metrics. Finsler metrics with vanishing Douglas curvature are called *Douglas metrics*. Douglas metrics form a rich class of Finsler metrics including locally projectively flat Finsler metrics.

Randers metrics are an important class of Finsler metrics, which are introduced by a physicist G. Randers in 1941. A Randers metric is of the form $F=\alpha+\beta$, where α is a Riemannian metric and β is a 1-form. However, it can also be expressed in the following navigation form

$$F=\frac{\sqrt{(1-b^2)\alpha^2+\beta^2}}{1-b^2}+\frac{\beta}{1-b^2}.$$

It is well-known that a Randers metric is a Douglas metric if and only if β is closed for both of the above expressions [1] . As a generalization of Randers metrics, (α,β)-metrics are also defined by a a Riemannian metric and a 1-form and given in the form

$$F=\alpha\phi(\frac{\beta}{\alpha}),$$

where ϕ is a smooth function and satisfies two additional conditions. In 2009, B. Li, Y. Shen and Z. Shen gave a characterization of Douglas (α,β)-metrics with dimension $n\geq 3$ [5]. Recently, C. Yu gave a more clear characterization. *If $F=\alpha\phi(\frac{\beta}{\alpha})$ is a non-trivial Douglas metric, then after some special deformations, α will turn to be another Riemannian metric $\bar{\alpha}$ and β to be another 1-form $\bar{\beta}$ such that $\bar{\beta}$ is close and conformal with respect to $\bar{\alpha}$, i.e., $\bar{b}_{i|j}=c(x)\bar{\alpha}_{ij}$, where $c(x)\neq 0$ is a scalar function on the manifold. In this case, F can be reexpressed as the form $F=\bar{\alpha}\phi(\bar{b}^2,\frac{\bar{\beta}}{\bar{\alpha}})$* [14].

In fact, many famous Douglas metrics can be also expressed in the following form

$$F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big), \tag{1.1}$$

where α is a Riemannian metric, β is a 1-form, $b:=\|\beta_x\|_\alpha$ and $\phi(b^2,s)$ is a smooth function. Finsler metrics in this form are called general (α,β)-metrics [15]. If $\phi=\phi(s)$ is independent of b^2, then $F=\alpha\phi(\frac{\beta}{\alpha})$ is a

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(α, β)-metric. If $\alpha = |y|$, $\beta = \langle x, y\rangle$, then $F = |y|\phi(|x|^2, \frac{\langle x,y\rangle}{|y|})$ is the so-called spherically symmetric Finsler metrics [8]. Moreover, general (α, β)-metrics include part of Bryant's metrics [2, 15] and part of fourth root metrics [6]. Besides Randers metrics, square metrics can be expressed in the following form

$$F = \frac{(\sqrt{(1-b^2)\alpha^2 + \beta^2} + \beta)^2}{(1-b^2)^2\sqrt{(1-b^2)\alpha^2 + \beta^2}},$$

It has been shown that F is a non-trivial Douglas square metrics if and only if

$$b_{i|j} = ca_{ij},$$

where $c = c(x) \neq 0$ is a scalar function on M [14].

In this paper, we mainly study general (α, β)-metrics with vanishing Douglas curvature. Firstly, a characterization equation for such metrics to be Douglas metrics under a suitable condition is given (Theorem 1.1). By solving this equation, we obtain all general (α, β)-metrics with vanishing Douglas curvature under certain condition(Theorem 1.2). At last, we explicitly construct some new examples (see Section 6).

Here, we will assume that β is closed and conformal with respect to α, i.e. (1.2) holds. According to the relate discussions for Douglas (α, β)-metrics [3, 5, 7, 8, 14], we believe that the assumption here is reasonable and appropriate.

The main results are given below.

Theorem 1.1. *Let $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ be a non-Riemannian general (α, β)-metric on an n-dimensional manifold M. Suppose that β satisfies*

$$b_{i|j} = ca_{ij}, \tag{1.2}$$

where $c = c(x) \neq 0$ is a scalar function on M and $b_{i|j}$ is the covariant derivation of β with respect to α. Then F is a Douglas metric if and only if the following PDE holds

$$\phi_{22} - 2(\phi_1 - s\phi_{12}) = \{f(b^2) + g(b^2)s^2\}\{\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\} \tag{1.3}$$

where $f(x)$ and $g(x)$ are two arbitrary differentiable functions.

Note that ϕ_1 means the derivation of ϕ with respect to the first variable b^2.

It should be pointed out that if the scalar function $c(x) = 0$, then according to Proposition 3.1, $D^i{}_{jkl} = 0$, namely, $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ is a Douglas metric for any function $\phi(b^2, s)$. So it will be regarded as a trivial case.

By solving equation (1.3), we have the following result

Theorem 1.2. *Let $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ be a non-Riemannian general (α, β)-metric on an n-dimensional manifold M. Suppose that β satisfies (1.2). Then the general solution of (1.3) is given by*

$$\phi = s\left(h(b^2) - \int \frac{\Phi\big(\eta(b^2, s)\big)}{s^2\sqrt{b^2 - s^2}}ds\right), \tag{1.4}$$

where

$$\eta(b^2, s) := \frac{b^2 - s^2}{e^{\int (f+gb^2)db^2} - (b^2 - s^2)\int g e^{\int (f+gb^2)db^2} db^2}, \tag{1.5}$$

where f, g and h are arbitrary smooth functions of b^2. Φ is an arbitrary smooth function of η. Moreover, the corresponding general (α, β)-metrics of (1.4) are of Douglas type.

Remark: If the general (α, β)-metrics $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ given by (1.4) are regular Finsler metrics, then (1.4) should satisfy Lemma 5.1.

2 Preliminaries

Let F be a Finsler metric on an n-dimensional manifold M and G^i be the geodesic coefficients of F, which are defined by

$$G^i = \frac{1}{4} g^{il}\left\{[F^2]_{x^k y^l} y^k - [F^2]_{x^l}\right\},$$

where $(g^{ij}) := \left(\frac{1}{2}[F^2]_{y^i y^j}\right)^{-1}$. For a Riemannian metric, the spray coefficients are determined by its Christoffel symbols as $G^i(x,y) = \frac{1}{2}\Gamma^i_{jk}(x) y^j y^k$.

By definition, a general (α,β)-metric is given by (1.1), where $\phi(b^2,s)$ is a positive smooth function defined on the domain $|s| \le b < b_o$ for some positive number (maybe infinity) b_o. Then the function $F = \alpha\phi(b^2,s)$ is a Finsler metric for any Riemannian metric $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and any 1-form $\beta = b_i(x)y^i$ if and only if $\phi(b^2,s)$ satisfies

$$\phi - s\phi_2 > 0, \quad \phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{2.1}$$

when $n \ge 3$ or

$$\phi - s\phi_2 + (b^2 - s^2)\phi_{22} > 0, \tag{2.2}$$

when $n = 2$ [15].

Let $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ and $\beta = b_i(x)y^i$. Denote the coefficients of the covariant derivative of β with respect to α by $b_{i|j}$, and let

$$r_{ij} = \tfrac{1}{2}(b_{i|j} + b_{j|i}),\ s_{ij} = \tfrac{1}{2}(b_{i|j} - b_{j|i}),\ r_{00} = r_{ij}y^i y^j,\ s^i{}_0 = a^{ij}s_{jk}y^k,$$
$$r_i = b^j r_{ji},\ s_i = b^j s_{ji},\ r_0 = r_i y^i,\ s_0 = s_i y^i,\ r^i = a^{ij} r_j,\ s^i = a^{ij}s_j,\ r = b^i r_i,$$

where $(a^{ij}) := (a_{ij})^{-1}$ and $b^i := a^{ij}b_j$. It is easy to see that β is closed if and only if $s_{ij} = 0$.

According to [15], the spray coefficients G^i of a general (α,β)-metric $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ are related to the spray coefficients ${}^\alpha G^i$ of α and given by

$$\begin{aligned} G^i \ =\ & {}^\alpha G^i + \alpha Q s^i{}_0 + \left\{\Theta(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Omega(r_0 + s_0)\right\}\frac{y^i}{\alpha} \\ & + \left\{\Psi(-2\alpha Q s_0 + r_{00} + 2\alpha^2 R r) + \alpha\Pi(r_0 + s_0)\right\} b^i - \alpha^2 R(r^i + s^i), \end{aligned} \tag{2.3}$$

where

$$Q = \frac{\phi_2}{\phi - s\phi_2}, \quad R = \frac{\phi_1}{\phi - s\phi_2},$$

$$\Theta = \frac{(\phi - s\phi_2)\phi_2 - s\phi\phi_{22}}{2\phi\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Psi = \frac{\phi_{22}}{2\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)},$$

$$\Pi = \frac{(\phi - s\phi_2)\phi_{12} - s\phi_1\phi_{22}}{(\phi - s\phi_2)\big(\phi - s\phi_2 + (b^2 - s^2)\phi_{22}\big)}, \quad \Omega = \frac{2\phi_1}{\phi} - \frac{s\phi + (b^2 - s^2)\phi_2}{\phi}\Pi.$$

In the following, we will introduce an important projective invariant.

Definition 2.1. [11] Let

$$D^i_{jkl} = \frac{\partial^3}{\partial y^j \partial y^k \partial y^l}\left(G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i\right), \tag{2.4}$$

where G^i are the spray coefficients of F. The tensor $D := D^i_{jkl}\partial_i \otimes dx^j \otimes dx^k \otimes dx^l$ is called *Douglas tensor.* A Finsler metric is called *a Douglas metric* if the Douglas tensor vanishes.

We require the following result in Section 6, the proof which is omitted.

Lemma 2.2. *Let*

$$I_n := \int s^{-2}(b^2-s^2)^{\frac{n-1}{2}}ds, \tag{2.5}$$

then for any natural number $n \geq 1$, we have

(a) $n = 2m$

$$\begin{aligned} I_{2m} = & \frac{(2m-1)!!}{(2m-2)!!}\frac{1}{s}\sum_{i=1}^{m-1}\frac{(2m-2-2i)!!}{(2m-2i+1)!!}\big(b^2\big)^{i-1}(b^2-s^2)^{\frac{2m-2i+1}{2}} \\ & -\frac{(2m-1)!!}{(2m-2)!!}\frac{1}{s}\big(b^2\big)^{m-1}\big[(b^2-s^2)^{\frac{1}{2}}+s\arctan\frac{s}{\sqrt{b^2-s^2}}\big]+C_1. \end{aligned} \tag{2.6}$$

(b) $n = 2m+1$

$$I_{2m+1} = \frac{(2m)!!}{(2m-1)!!}\frac{1}{s}\left[\sum_{i=1}^{m}\frac{(2m-2i-1)!!}{(2m-2i+2)!!}\big(b^2\big)^{i-1}(b^2-s^2)^{m-i+1}-\big(b^2\big)^m\right]+C_2, \tag{2.7}$$

where C_1 and C_2 are arbitrary constants.

3 Douglas curvature of general (α,β)-metrics

In this section, we will compute the Douglas curvature of a general (α,β)-metric.

Proposition 3.1. *Let $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ be a general (α,β)-metric on an n-dimensional manifold M. Suppose that β satisfies (1.2), then the Douglas curvature of F is given by*

$$\begin{aligned} D^i{}_{jkl} = & \frac{c}{\alpha}\left\{[(T-sT_2)a_{kl}+T_{22}b_lb_k]\,\delta^i{}_j+\frac{1}{\alpha^2}\left[\frac{s}{\alpha}(3T_{22}+sT_{222})y_ly_j-(T_{22}+sT_{222})b_ly_j\right]b_ky^i\right\}(k\to l\to j\to k) \\ & -\frac{c}{\alpha^2}\left\{sT_{22}\left[(y_kb_l+y_lb_k)\delta^i{}_j+a_{jl}b_ky^i\right]+\frac{1}{\alpha}(T-sT_2-s^2T_{22})(y_l\delta^i{}_j+a_{lj}y^i)y_k\right\}(k\to l\to j\to k) \\ & +\frac{c}{\alpha^2}\left[\frac{1}{\alpha^3}(3T-3sT_2-6s^2T_{22}-s^3T_{222})y_ky_jy_l+T_{222}b_lb_kb_j\right]y^i \\ & +\frac{c}{\alpha}\left[(H_2-sH_{22})(b_j-\frac{s}{\alpha}y_j)a_{kl}-\frac{1}{\alpha^2}(H_2-sH_{22}-s^2H_{222})b_ly_jy_k-\frac{sH_{222}}{\alpha}b_kb_ly_j\right]b^i(k\to l\to j\to k) \\ & +\frac{c}{\alpha}\left[\frac{s}{\alpha^3}(3H_2-3sH_{22}-s^2H_{222})y_jy_ky_l+H_{222}b_lb_kb_j\right]b^i, \end{aligned} \tag{3.1}$$

where $y_i := a_{ij}y^j$ and $b^i := a^{ij}b_j$, $c = c(x) \neq 0$ is a scalar function on M.

$$\begin{aligned} T : = & -\frac{1}{n+1}[2sH+(b^2-s^2)H_2], & (3.2) \\ H : = & \frac{\phi_{22}-2(\phi_1-s\phi_{12})}{2\big[\phi-s\phi_2+(b^2-s^2)\phi_{22}\big]}. & (3.3) \end{aligned}$$

Proof. By (1.2), we have

$$r_{00} = c\alpha^2, r_0 = c\beta, r = cb^2, r^i = cb^i, s^i{}_0 = 0, s_0 = 0, s^i = 0. \tag{3.4}$$

Substituting (3.4) into (2.3) yields

$$\begin{aligned} G^i = & {}^\alpha G^i + c\alpha\left\{\Theta(1+2Rb^2)+s\Omega\right\}y^i + c\alpha^2\left\{\Psi(1+2Rb^2)+s\Pi-R\right\}b^i \\ = & {}^\alpha G^i + c\alpha Ey^i + c\alpha^2Hb^i, \end{aligned} \tag{3.5}$$

where

$$E := \frac{\phi_2 + 2s\phi_1}{2\phi} - H\frac{s\phi + (b^2 - s^2)\phi_2}{\phi}.$$

Note that

$$\alpha_{y^i} = \frac{y_i}{\alpha}, \quad s_{y^i} = \frac{\alpha b_i - s y_i}{\alpha^2}, \tag{3.6}$$

where $y_i := a_{ij} y^j$.

$$\frac{\partial G^m}{\partial y^m} = \frac{\partial\, {}^\alpha G^m}{\partial y^m} + c\alpha[(n+1)E + 2sH + (b^2 - s^2)H_2], \tag{3.7}$$

where we take Einstein summation convention. By (3.5) and (3.7), we have

$$G^i - \frac{1}{n+1}\frac{\partial G^m}{\partial y^m} y^i = {}^\alpha G^i - \frac{1}{n+1}\frac{\partial\, {}^\alpha G^m}{\partial y^m} y^i + c\alpha(T y^i + \alpha H b^i). \tag{3.8}$$

Put

$$W^i := \alpha T y^i + \alpha^2 H b^i. \tag{3.9}$$

Differentiating (3.9) with respect to y^j yields

$$\frac{\partial W^i}{\partial y^j} = \alpha T \delta^i{}_j + (T\alpha_{y^j} + \alpha T_2 s_{y^j}) y^i + \big\{[\alpha^2]_{y^j} H + \alpha^2 H_2 s_{y^j}\big\} b^i. \tag{3.10}$$

Differentiating (3.10) with respect to y^k yields

$$\begin{aligned}\frac{\partial^2 W^i}{\partial y^j \partial y^k} \;=\; & \big[(T\alpha_{y^k} + \alpha T_2 s_{y^k})\delta^i{}_j + T_2 s_{y^k}\alpha_{y^j} y^i + H_2[\alpha^2]_{y^j} s_{y^k} b^i\big](k \leftrightarrow j) \\ & + \big(T\alpha_{y^j y^k} + \alpha T_{22} s_{y^k} s_{y^j} + \alpha T_2 s_{y^j y^k}\big) y^i \\ & + \big\{[\alpha^2]_{y^j y^k} H + \alpha^2 H_{22} s_{y^k} s_{y^j} + \alpha^2 H_2 s_{y^j y^k}\big\} b^i, \end{aligned} \tag{3.11}$$

where $k \leftrightarrow j$ denotes symmetrization. Therefore, it follows from (3.11) that

$$\begin{aligned}\frac{\partial^3 W^i}{\partial y^j \partial y^k \partial y^l} \;=\; & \big[T_2(\alpha_{y^k} s_{y^l} + \alpha_{y^l} s_{y^k} + \alpha s_{y^k y^l}) + T\alpha_{y^k y^l} + \alpha T_{22} s_{y^l} s_{y^k}\big]\delta^i{}_j (k \to l \to j \to k) \\ & + \big[T_2(s_{y^k}\alpha_{y^j y^l} + \alpha_{y^k} s_{y^j y^l}) + T_{22}(\alpha_{y^k} s_{y^j} + \alpha s_{y^k y^j}) s_{y^l}\big] y^i (k \to l \to j \to k) \\ & + \{H_2\big([\alpha^2]_{y^k y^l} s_{y^j} + [\alpha^2]_{y^k} s_{y^j y^l}\big) + H_{22}\big([\alpha^2]_{y^k} s_{y^l} s_{y^j} + \alpha^2 s_{y^k y^l} s_{y^j}\big)\} b^i (k \to l \to j \to k) \\ & + \big(T\alpha_{y^j y^k y^l} + \alpha T_{222} s_{y^j} s_{y^k} s_{y^l} + \alpha T_2 s_{y^j y^k y^l}\big) y^i \\ & + \big\{H[\alpha^2]_{y^j y^k y^l} + \alpha^2 H_{222} s_{y^j} s_{y^k} s_{y^l} + \alpha^2 H_2 s_{y^j y^k y^l}\big\} b^i, \end{aligned} \tag{3.12}$$

where $k \to l \to j \to k$ denotes cyclic permutation. It follows from (3.6) that

$$[\alpha^2]_{y^l} \;=\; 2y_l, \quad [\alpha^2]_{y^l y^j} = 2a_{lj}, \quad [\alpha^2]_{y^l y^j y^k} = 0, \tag{3.13}$$

$$\alpha_{y^l y^j} \;=\; \frac{1}{\alpha}\big(a_{lj} - \frac{y_l}{\alpha}\frac{y_j}{\alpha}\big), \quad \alpha_{y^l y^j y^k} = -\frac{1}{\alpha^3}[a_{kl} y_j (k \to l \to j \to k) - \frac{3}{\alpha^2} y_l y_j y_k], \tag{3.14}$$

$$s_{y^l y^j} \;=\; -\frac{1}{\alpha^2}\big[s a_{lj} + \frac{1}{\alpha}(b_l y_j + b_j y_l) - \frac{3s}{\alpha^2} y_l y_j\big], \tag{3.15}$$

$$s_{y^l y^j y^k} \;=\; \frac{1}{\alpha^5}\big\{[\alpha(3s y_j - \alpha b_j) a_{lk} + 3b_k y_l y_j](k \to l \to j \to k) - \frac{15s}{\alpha} y_k y_l y_j\big\}. \tag{3.16}$$

Plugging (3.13), (3.14), (3.15) and (3.16) into (3.12) yields

$$\begin{aligned}\frac{\partial^3 W^i}{\partial y^j\partial y^k\partial y^l} &= \frac{1}{\alpha}\Big\{\big[(T-sT_2)a_{kl}+T_{22}b_lb_k\big]\delta^i{}_j+\frac{1}{\alpha^2}\big[\frac{s}{\alpha}(3T_{22}+sT_{222})y_l-(T_{22}+sT_{222})b_l\big]y_jb_ky^i\Big\}(k\to l\to j\to k)\\ &-\frac{1}{\alpha^2}\Big\{sT_{22}\big[(y_kb_l+y_lb_k)\delta^i{}_j+a_{jl}b_ky^i\big]+\frac{1}{\alpha}(T-sT_2-s^2T_{22})(y_l\delta^i_j+a_{jl}y^i)y_k\Big\}(k\to l\to j\to k)\\ &+\frac{1}{\alpha^2}\Big[\frac{1}{\alpha^3}(3T-3sT_2-6s^2T_{22}-s^3T_{222})y_ky_jy_l+T_{222}b_lb_kb_j\Big]y^i\\ &+\frac{1}{\alpha}\Big[(H_2-sH_{22})(b_j-\frac{s}{\alpha}y_j)a_{kl}-\frac{1}{\alpha^2}(H_2-sH_{22}-s^2H_{222})b_ly_jy_k-\frac{sH_{222}}{\alpha}b_kb_ly_j\Big]b^i(k\to l\to j\to k)\\ &+\frac{1}{\alpha}\Big[\frac{s}{\alpha^3}(3H_2-3sH_{22}-s^2H_{222})y_jy_ky_l+H_{222}b_lb_kb_j\Big]b^i, \qquad (3.17)\end{aligned}$$

It follows from ${}^{\alpha}G^i(x,y)=\frac{1}{2}\Gamma^i_{jk}(x)y^jy^k$ that

$$\frac{\partial^3}{\partial y^j\partial y^k\partial y^l}\big[{}^{\alpha}G^i-\frac{1}{n+1}\frac{\partial{}^{\alpha}G^m}{\partial y^m}y^i\big]=0 \qquad (3.18)$$

By (2.4), (3.8), (3.9), (3.17) and (3.18), we obtain (3.1).

□

4 Proof of Theorem 1.1

In this section, we mainly prove Theorem 1.1. Firstly, we give the following Lemma.

Lemma 4.1. *Suppose that β satisfies (1.2), then a general (α,β)-metric $F=\alpha\phi\Big(b^2,\frac{\beta}{\alpha}\Big)$ is a Douglas metric if and only if $H_2-sH_{22}=0$, where H is given by (3.3).*

Proof. Suppose that a general (α,β)-metric F is a Douglas metric, then the Douglas curvature of F vanishes, i.e, $D^i{}_{jkl}=0$. From (1.2) and (3.1), it follows that both rational and irrational parts of $D^i{}_{jkl}=0$ should vanish, i.e,

$$\begin{aligned}&\alpha^4\big\{[(T-sT_2)a_{kl}+T_{22}b_lb_k]\delta^i{}_j+(H_2-sH_{22})a_{kl}b_jb^i+\frac{1}{3}H_{222}b_lb_kb_jb^i\big\}(k\to l\to j\to k)\\ &-\alpha^2\big[(T_{22}+sT_{222})b_lb_ky_jy^i+(T-sT_2-s^2T_{22})(y_l\delta^i{}_j+a_{jl}y^i)y_k+(H_2-sH_{22}-s^2H_{222})b_ly_jy_kb^i\big](k\to l\to j\to k)\\ &+(3T-3sT_2-6s^2T_{22}-s^3T_{222})y_ky_jy_lb^i=0, \qquad (4.1)\\ &\alpha^2\big\{\frac{1}{3}T_{222}b_lb_kb_jy^i-sT_{22}[(y_kb_l+y_lb_k)\delta^i{}_j+a_{jl}b_ky^i]-s[(H_2-sH_{22})a_{kl}y_j+H_{222}b_kb_ly_j]b^i\big\}(k\to l\to j\to k)\\ &+s(3T_{22}+sT_{222})y_ly_jb_ky^i(k\to l\to j\to k)+s(3H_2-3sH_{22}-s^2H_{222})y_jy_ky_lb^i=0, \qquad (4.2)\end{aligned}$$

where H and T are given by (3.3) and (3.2), respectively. For $s\neq 0$, multiplying (4.2) by $y^jy^ky^l$ yields

$$T_{222}y^i-\alpha H_{222}b^i=0. \qquad (4.3)$$

By (4.3), it is easy to see that

$$T_{222}=0,\quad H_{222}=0. \qquad (4.4)$$

Inserting (4.4) into (4.2) yields

$$\begin{aligned}&\alpha^2s\big\{T_{22}[(y_kb_l+y_lb_k)\delta^i{}_j+a_{jl}b_ky^i]+(H_2-sH_{22})a_{kl}y_jb^i\big\}(k\to l\to j\to k)\\ &-3sT_{22}y_ly_jb_ky^i(k\to l\to j\to k)-3s(H_2-sH_{22})y_jy_ky_lb^i=0. \qquad (4.5)\end{aligned}$$

Multiplying (4.5) by $b^jb^kb^l$ yields

$$b^2(b^2-3s^2)T_{22}y^i+\alpha s[2b^2T_{22}+(b^2-s^2)(H_2-sH_{22})]b^i=0. \qquad (4.6)$$

It follows from (4.6) that

$$T_{22} = 0, \quad H_2 - sH_{22} = 0. \tag{4.7}$$

Plugging (4.4) and (4.7) into (4.1), we have

$$(T - sT_2)\big\{\alpha^2[\alpha^2 a_{kl}\delta^i{}_j - (y_l\delta^i{}_j + a_{jl}y^i)y_k](k \to l \to j \to k) + 3y_k y_j y_l y^i\big\} = 0 \tag{4.8}$$

Multiplying (4.8) by $b^j b^k b^l$ yields

$$(T - sT_2)(b^2 - s^2)(\alpha b^i - sy^i) = 0. \tag{4.9}$$

By (4.9), we have

$$T - sT_2 = 0. \tag{4.10}$$

By (3.2), we obtain

$$T - sT_2 = -\frac{1}{n+1}(b^2 - s^2)(H_2 - sH_{22}). \tag{4.11}$$

By (4.11), it is easy to see that the second equality of (4.7) implies (4.10).

Conversely, suppose that the second equality of (4.7) holds, it follows from (4.11) that (4.10) holds. Moreover,

$$T_{22} = 0, \quad T_{222} = 0, \quad H_{222} = 0. \tag{4.12}$$

Plugging the second equality of (4.7), (4.10) and (4.12) into (3.1), we have $D^i{}_{jkl} = 0$. Hence, general (α, β)-metric $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ is a Douglas metric. □

Proof of Theorem 1.1. By Lemma 4.1, we obtain

$$H = \frac{1}{2}[f(b^2) + g(b^2)s^2], \tag{4.13}$$

where f and g are two arbitrary smooth functions of b^2. By (3.3) and (4.13), we will complete the proof of Theorem 1.1. □

By taking $f = 0$ and $g = 0$, we obtain the following result [13]

Corollary 4.2. *Let $F = \alpha\phi\left(b^2, \frac{\beta}{\alpha}\right)$ be a Finsler metric. Suppose that β satisfies (1.2). Then F is projectively equivalent to α if and only if $\phi(b^2, s)$ satisfies*

$$\phi_{22} - 2(\phi_1 - s\phi_{12}) = 0.$$

5 General (α, β)-metrics with vanishing Douglas curvature

Proof of Theorem 1.2. Note that $(\phi - s\phi_2)_2 = -s\phi_{22}$. Therefore, (1.3) is changed to the following form

$$2(\phi - s\phi_2)_1 + \frac{1}{s}[1 - (f + gs^2)(b^2 - s^2)](\phi - s\phi_2)_2 + (f + gs^2)(\phi - s\phi_2) = 0. \tag{5.1}$$

Put

$$\psi := (\phi - s\phi_2)\sqrt{b^2 - s^2}. \tag{5.2}$$

Then

$$\psi_1 = (\phi - s\phi_2)_1\sqrt{b^2 - s^2} + \frac{1}{2\sqrt{b^2 - s^2}}(\phi - s\phi_2), \tag{5.3}$$

$$\psi_2 = (\phi - s\phi_2)_2\sqrt{b^2 - s^2} - \frac{s}{\sqrt{b^2 - s^2}}(\phi - s\phi_2). \tag{5.4}$$

It follows from (5.3) and (5.4) that (5.1) is equivalent to

$$\psi_1 + \frac{1}{2s}\big[1 - (f + gs^2)(b^2 - s^2)\big]\psi_2 = 0. \tag{5.5}$$

The characteristic equation of PDE (5.5) is

$$\frac{db^2}{1} = \frac{ds}{\frac{1}{2s}\big[1 - (f + gs^2)(b^2 - s^2)\big]} \tag{5.6}$$

(5.6) is equivalent to

$$2s\frac{ds}{db^2} = 1 - (f + gs^2)(b^2 - s^2). \tag{5.7}$$

Set

$$\chi(b^2) = s^2(b^2) - b^2. \tag{5.8}$$

Plugging (5.8) into (5.7) yields

$$\frac{d\chi}{db^2} = (f + gb^2)\chi + g\chi^2.$$

This is a Bernoulli equation which can be rewritten as

$$\frac{d}{db^2}(\frac{1}{\chi}) = -(f + gb^2)\frac{1}{\chi} - g.$$

This is a linear 1-order ODE of $\frac{1}{\chi}$. One can easily get its solution

$$\frac{1}{\chi} = -e^{-\int (f+gb^2)db^2}\big[c + \int ge^{\int (f+gb^2)db^2}db^2\big], \tag{5.9}$$

where c is an arbitrary constant. By (5.8) and (5.9), the independent integral of (5.6) is

$$\frac{b^2 - s^2}{e^{\int (f+gb^2)db^2} - (b^2 - s^2)\int ge^{\int (f+gb^2)db^2}db^2} = \frac{1}{c}.$$

Hence the solution of (5.5) is

$$\psi = \Phi\left(\frac{b^2 - s^2}{e^{\int (f+gb^2)db^2} - (b^2 - s^2)\int ge^{\int (f+gb^2)db^2}db^2}\right), \tag{5.10}$$

where Φ is any continuously differentiable function. By (5.2) and (5.10), we have

$$\phi - s\phi_2 = \Phi\left(\frac{b^2 - s^2}{e^{\int (f+gb^2)db^2} - (b^2 - s^2)\int ge^{\int (f+gb^2)db^2}db^2}\right)\frac{1}{\sqrt{b^2 - s^2}}. \tag{5.11}$$

Let $\phi = s\varphi$, then we have

$$\phi - s\phi_2 = -s^2\varphi_2. \tag{5.12}$$

By (5.11) and (5.12), we obtain

$$\varphi = h(b^2) - \int \frac{\Phi\big(\eta(b^2, s)\big)}{s^2\sqrt{b^2 - s^2}}ds, \tag{5.13}$$

where $h(x)$ is an arbitrary smooth function and $\eta(b^2, s)$ is given by (1.5). Hence, by $\phi = s\varphi$, we get (1.4). $\square$

In the following, we will give necessary and sufficient conditions for a general (α, β)-metric with vanishing Douglas curvature to be a Finsler metric.

Lemma 5.1. *Let $F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$ be a general (α, β)-metric on an n-dimensional manifold M, where ϕ is given by (1.4). Then F is a Finsler metric if and only if*

$$\frac{\Phi}{\sqrt{b^2 - s^2}} > 0, \quad -\frac{\sqrt{b^2 - s^2}}{s}\Phi_2 > 0. \tag{5.14}$$

when $n \geq 3$ or

$$-\frac{\sqrt{b^2 - s^2}}{s}\Phi_2 > 0. \tag{5.15}$$

when $n = 2$.

Proof. Note that $-s\phi_{22} = (\phi - s\phi)_2$. By (2.1), (2.2) and (5.11), we will get (5.14) and (5.15). $\square$

6 Some new examples

In this section, we will explicitly construct some new examples .

Example 6.1. Take $g = 0$ and $\Phi(\eta(b^2, s)) = (b^2 - s^2)^{\frac{m}{2}} e^{-\int f db^2}$, then for any natural number $m \geq 1$, parts of the solutions of (1.4) are given by

(a) $m = 2l$

$$\begin{aligned}\phi(b^2, s) \;=\; & \tilde{h}_1(b^2)s - e^{-\int f db^2}\frac{(2l-1)!!}{(2l-2)!!}\{\sum_{i=1}^{l-1}\frac{(2l-2-2i)!!}{(2l-2i+1)!!}\big(b^2\big)^{i-1}\big(b^2 - s^2\big)^{\frac{2l-2i+1}{2}} \\ & -\big(b^2\big)^{l-1}\big[(b^2 - s^2)^{\frac{1}{2}} + s\arctan\frac{s}{\sqrt{b^2 - s^2}}\big]\},\end{aligned}$$

(b) $m = 2l + 1$

$$\phi(b^2, s) \;=\; \tilde{h}_2(b^2)s - e^{-\int f db^2}\frac{(2l)!!}{(2l-1)!!}\left[\sum_{i=1}^{l}\frac{(2l-2i-1)!!}{(2l-2i+2)!!}\big(b^2\big)^{i-1}(b^2 - s^2)^{l-i+1} - \big(b^2\big)^{l}\right],$$

where $\tilde{h}_1$, $\tilde{h}_2$ and f are any smooth functions of b^2 such that ϕ is positive. Moreover, the corresponding general (α, β)-metrics

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$$

are of Douglas type.

Note that we have made use of Lemma 2.2.

Example 6.2. Take $g = 0$, $f = \frac{\mu^2 + \varepsilon\xi}{\varepsilon + (\mu^2 + \varepsilon\xi)b^2}$ and $\Phi(\eta(b^2, s)) = \varepsilon\sqrt{\frac{\eta}{1 - \mu^2\eta}}$, then parts of the solutions of (1.4) are given by

$$\phi(b^2, s) = \tilde{h}(b^2)s + \frac{\sqrt{\varepsilon + \varepsilon\xi b^2 + \mu^2 s^2}}{1 + \xi b^2}, \tag{6.1}$$

where $\tilde{h}$ is a smooth function of b^2 and μ, ε, ξ are constants such that ϕ is positive. Moreover, the corresponding general (α, β)-metrics

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$$

are of Douglas type.

Remark: Especially, take $\tilde{h}(b^2) = \frac{\mu}{1+\xi b^2}$ in (6.1), we have

$$\phi(b^2, s) = \frac{\sqrt{\varepsilon + \varepsilon\xi b^2 + \mu^2 s^2}}{1 + \xi b^2} + \frac{\mu s}{1 + \xi b^2}. \tag{6.2}$$

(1) Take $\alpha = |y|$ and $\beta = \langle x, y\rangle$, then the corresponding general (α, β)-metrics of (6.2)

$$F = \frac{\sqrt{\varepsilon(1 + \xi|x|^2) + \mu^2\langle x, y\rangle^2}}{1 + \xi|x|^2} + \frac{\mu\langle x, y\rangle^2}{1 + \xi|x|}$$

are of Douglas type. In fact, they belong to spherically symmetric Douglas metrics, too. Moreover, when $\varepsilon = 1$, $\xi = -1$ and $\mu = \pm 1$, F is just the Funk metric.

(2) Take $\alpha = |y|$ and $\beta = \langle x, y\rangle + \langle a, y\rangle$, where a is a constant vector, then the corresponding general (α, β)-metrics of (6.2)

$$F = \frac{\sqrt{(1 - |x|^2 - 2\langle a, x\rangle - |a|^2)|y|^2 + (\langle x, y\rangle + \langle a, x\rangle)^2}}{1 - |x|^2 - 2\langle a, x\rangle - |a|^2} \pm \frac{\langle x, y\rangle + \langle a, x\rangle}{1 - |x|^2 - 2\langle a, x\rangle - |a|^2}$$

are of Douglas type(See Example 8.1 in [16]). Actually, they are just the generalized Funk metrics expressed in some other local coordinate system.

Example 6.3. Take $f = g = 0$ and $\Phi(\eta(b^2, s)) = (1 + \eta)\sqrt{\eta}$, then parts of the solutions of (1.4) are given by

$$\phi(b^2, s) = \tilde{h}(b^2)s + 1 + b^2 + s^2, \tag{6.3}$$

where $\tilde{h}$ is a smooth function of b^2 such that ϕ is positive. Moreover, the corresponding general (α, β)-metrics

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$$

are of Douglas type.

Remark: Take $\tilde{h}(b^2) = 2\sqrt{1 + b^2}$ in (6.3), $\alpha = \frac{\sqrt{(1+\mu|x|^2)|y|^2 - \mu\langle x,y\rangle^2}}{1+\mu|x|^2}$ and $\beta = \frac{\langle x,y\rangle}{(1+\mu|x|^2)^{\frac{3}{2}}}$, where μ is a constant. We obtain Example 4.3 given in [15], namely

$$F = \frac{(\sqrt{1 + (1 + \mu)|x|^2}\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2} + \langle x, y\rangle)^2}{(1 + \mu|x|^2)^2\sqrt{(1 + \mu|x|^2)|y|^2 - \mu\langle x, y\rangle^2}}.$$

In particular, F is just the Berwald's metric when $\mu = -1$.

Example 6.4. Take $f = g = 0$ and $\Phi(\eta(b^2, s)) = \frac{\sqrt{\eta}}{(1-\eta)^{\frac{3}{2}}}$, then parts of the solutions of (1.4) are given by

$$\phi(b^2, s) = \tilde{h}(b^2)s + \frac{1 - b^2 + 2s^2}{(1 - b^2)\sqrt{1 - b^2 + s^2}}, \tag{6.4}$$

where $\tilde{h}$ is a smooth function of b^2 such that ϕ is positive. Moreover, the corresponding general (α, β)-metrics

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$$

are of Douglas type.

Remark: Take $\tilde{h}(b^2) = \mp\frac{2}{(1-b^2)^2}$ in (6.4), $\alpha = |y|$ and $\beta = \langle x, y\rangle + \langle a, y\rangle$, where a is a constant vector, the corresponding general (α, β)-metrics of (6.4)

$$F = \frac{\{\sqrt{(1 - |x|^2 - 2\langle a, x\rangle - |a|^2)|y|^2 + (\langle x, y\rangle + \langle a, y\rangle)^2} \mp (\langle x, y\rangle + \langle a, y\rangle)\}^2}{(1 - |x|^2 - 2\langle a, x\rangle - |a|^2)^2\sqrt{(1 - |x|^2 - 2\langle a, x\rangle - |a|^2)|y|^2 + (\langle x, y\rangle + \langle a, y\rangle)^2}}$$

are of Douglas type(See Example 8.2 in [16]). Actually, they are just the generalized Berwald's metrics expressed in some other local coordinate system.

Example 6.5. Take $f = g = 0$ and $\Phi(\eta(b^2, s)) = \frac{1}{2}[\frac{1}{\sqrt{c-\eta}} - \frac{\varepsilon}{\sqrt{c-\varepsilon^2\eta}}]\sqrt{\eta}$, then parts of the solutions of (1.4) are given by

$$\phi(b^2, s) = \tilde{h}(b^2)s + \frac{1}{2}\left[\frac{\sqrt{c - b^2 + s^2}}{c - b^2} - \frac{\varepsilon\sqrt{c - \varepsilon^2(b^2 - s^2)}}{c - \varepsilon^2 b^2}\right], \tag{6.5}$$

where $c > 0$, $\varepsilon < 1$, $\tilde{h}$ is a smooth function of b^2 such that ϕ is positive. Moreover, the corresponding general (α, β)-metrics

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$$

are of Douglas type.

Remark: Take $\tilde{h}(b^2) = \frac{1}{2}\left(\frac{1}{c-b^2} - \frac{\varepsilon^2}{c-\varepsilon^2 b^2}\right)$ in (6.5), $\alpha = |y|$ and $\beta = \langle x, y\rangle + \langle a, y\rangle$, where a is a constant vector, the corresponding general (α, β)-metrics of (6.5)

$$\begin{aligned} F \;=\; & \frac{1}{2}\Bigg\{\frac{\sqrt{(c - |x|^2 - 2\langle a, x\rangle - |a|^2)|y|^2 + (\langle x, y\rangle + \langle a, y\rangle)^2} + \langle x, y\rangle + \langle a, y\rangle}{c - |x|^2 - 2\langle a, x\rangle - |a|^2} \\ & - \frac{\varepsilon\sqrt{[c - \varepsilon^2(|x|^2 + 2\langle a, x\rangle + |a|^2)]|y|^2 + \varepsilon^2(\langle x, y\rangle + \langle a, y\rangle)^2} + \varepsilon^2(\langle x, y\rangle + \langle a, y\rangle)}{c - \varepsilon^2(|x|^2 + 2\langle a, x\rangle + |a|^2)}\Bigg\} \end{aligned}$$

are of Douglas type. In particular, when $c = 1$ and $a = 0$, it is just Shen' metrics(see (39) in [12]). When $c = 1$, it is just the Example 8.4 in [16]. When $a = 0$, it is just a projectively flat spherically symmetric Finsler metrics with constant flag curvature -1 [9].

Example 6.6. Take $f = \lambda$, $g = \frac{\lambda^2}{1-\lambda b^2}$ and $\Phi(\eta(b^2, s)) = \sqrt{\eta}$, then parts of the solutions of (1.4) are given by

$$\phi(b^2, s) = \tilde{h}(b^2)s + \frac{\sqrt{(1 - \lambda b^2)(1 - 2\lambda b^2 + \lambda s^2)}}{1 - 2\lambda b^2}, \tag{6.6}$$

where λ is an arbitrary constant, $\tilde{h}$ is a smooth function of b^2 such that ϕ is positive. Moreover, the corresponding general (α, β)-metrics

$$F = \alpha\phi\Big(b^2, \frac{\beta}{\alpha}\Big)$$

are of Douglas type.

Remark: Take $\tilde{h}(b^2) = \frac{\sqrt{1-\lambda b^2}}{1-2\lambda b^2}$ in (6.6), $\alpha = |y|$ and $\beta = \langle x, y\rangle + \langle a, y\rangle$, where a is a constant vector, the corresponding general (α, β)-metrics of (6.6)

$$F = \frac{\sqrt{1 - \lambda(|x|^2 + 2\langle a, x\rangle + |a|^2)}}{1 - 2\lambda(|x|^2 + 2\langle a, x\rangle + |a|^2)}\{\sqrt{(1 - 2\lambda(|x|^2 + 2\langle a, x\rangle + |a|^2))|y|^2 + \lambda(\langle x, y\rangle + \langle a, y\rangle)^2} + \langle x, y\rangle + \langle a, y\rangle\}$$

are of Douglas type, but not locally projectively flat.

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